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VOLUMES FOR TWIST LINK CONE-MANIFOLDS

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ABSTRACT. Recently, the explicit volume formulae for hyperbolic cone-manifolds, whose underlying space is the 3-sphere and the singular set is the knot 4_1 and the links 5_1^2 and 6_2^2, have been obtained by the second named author and his collaborators. In this paper we explicitly find the hyperbolic volume for cone-manifolds with the link 6_3^2 as singular set. Trigonometric identities (Tangent, Sine and Cosine Rules) between complex lengths of singular components and cone angles are obtained for an infinite family of two-bridge links containing 5_1^2 and 6_3^2.

1. Introduction

Starting from Alexander's works, polynomial invariants have become a very convenient instrument for knot investigation. Several kinds of knots polynomials have been discovered in the last twenty years. Among these, we recall the Jones-, Kaufmann-, HOMFLY-, A-polynomials and others ([12], [3], [8]). These polynomials relate knot theory to algebra and algebraic geometry. Algebraic techniques are used to find the most important geometrical characteristics of knots, such as volume, length of shortest geodesics and others.

The explicit volume formulae for hyperbolic cone-manifolds, whose underlying space is the 3-sphere and the singular set is the knot 4_1 and the links 5_1^2 and 6_2^2, have been obtained in [17], [19] and [15].

The aim of our paper is to explicitly find the hyperbolic volume for cone-manifolds with the link 6_3^2 as singular set. In order to do this, we will introduce a family of hyperbolic cone-manifolds $W_p(\alpha, \beta)$, with the two-bridge links W_p, with slope $(4p+4)/(2p+1)$ as singular set, and α, β as cone angles.

Trigonometric identities (Tangent, Sine and Cosine Rules) between complex lengths of singular components and cone angles for $W_p(\alpha, \beta)$ are obtained. Then the Schläfli formula applies in order to find explicit hyperbolic volumes for cone-manifolds $W_2(\alpha, \beta)$.

In the present paper links and knots are considered as singular subsets of the three-sphere endowed by a Riemannian metric of negative constant curvature.

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2. Trigonometric identities for knots and links

(2.1) Cone-manifolds, complex distances and lengths. We start with the definition of cone-manifold modelled in a hyperbolic, spherical, or Euclidian structure.

Definition (2.1.1). A 3-dimensional *hyperbolic cone-manifold* is a Riemannian 3-dimensional manifold of constant negative sectional curvature with cone-type singularities along simple closed geodesics.

To each component of the singular set is associated a real number $n \geq 1$ such that the cone-angle around the component is $\alpha = 2\pi/n$. The concept of hyperbolic cone-manifold generalizes that of hyperbolic manifold, which appears in the partial case when all cone-angles are 2π. Hyperbolic cone-manifolds are also a generalization of hyperbolic 3-orbifolds, which arises when all associated numbers n are integers. Euclidean and spherical cone-manifolds are defined similarly.

In the present paper hyperbolic, spherical or Euclidean cone-manifolds C are considered whose underlying space is the three-dimensional sphere and the singular set $\Sigma = \Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^k$ is a link consisting of the components $\Sigma^j = \Sigma^j(\alpha_j)$, $j = 1, 2, \ldots, k$ with cone-angles $\alpha_1, \ldots, \alpha_k$ respectively.

We recall a few well-known facts from hyperbolic geometry.

Let $\mathbb{H}^3 = \{(z, \xi) \in \mathbb{C} \times \mathbb{R} : \xi > 0\}$ be the upper half space model of the 3 -dimensional hyperbolic space endowed by the Riemannian metric

$$ds^2 = \frac{dzd\overline{z} + d\xi^2}{\xi^2}.$$

We identify the group of orientation preserving isometries of $\mathbb{H}^3$ with the group $PSL(2, \mathbb{C})$, consisting of linear fractional transformations

$$A' \colon z \in \mathbb{C} \to \frac{az + b}{cz + d}\,.$$

By a canonical procedure, A' can be uniquely extended to an isometry of $\mathbb{H}^3$. We prefer to deal with the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ rather than the element $A' \in PSL(2, \mathbb{C})$. The matrix A is uniquely determined by the element A', up to a sign. In the following we will use the same letter A for both A and A', as long as this does not create confusion.

Let C be a hyperbolic cone-manifold with the singular set Σ. Then C defines a nonsingular but incomplete hyperbolic manifold $\mathcal{M} = C - \Sigma$. Denote by Φ the fundamental group of the manifold $\mathcal{M}$.

The hyperbolic structure of $\mathcal{M}$ defines, up to conjugation in $PSL(2, \mathbb{C})$, a holonomy homomorphism

$$\hat{h} \colon \Phi \to PSL(2, \mathbb{C}).$$

It is shown in [23] that the holonomy homomorphism of an orientable cone-manifold can be lifted to $SL(2, \mathbb{C})$ if all cone-angles are at most π. Denote by $h : \Phi \to SL(2, \mathbb{C})$ this lifting homomorphism. Choose an orientation on the link $\Sigma = \Sigma^1 \cup \Sigma^2 \cup \ldots \cup \Sigma^k$ and fix a meridian-longitude pair $\{m_j,\, l_j\}$ for each component $\Sigma^j = \Sigma^j(\alpha_j)$. Then the matrices $M_j = h(m_j)$ and $L_j = h(l_j)$ satisfy the following properties:

$$\mathrm{tr}\,(M_j) = 2\cos(\alpha_j/2), \quad M_j L_j = L_j M_j\,, \;\; j = 1, 2, \ldots, k\,.$$

Now we point out some definitions and results from the book [4]. A matrix $A \in SL(2, \mathbb{C})$ satisfying $\mathrm{tr}\,(A) = 0$ is called a (normalized) *line matrix*. We have by definition that $A^2 = -I$, where I is the identity matrix. Hence any line matrix determines a half-turn about a line in $\mathbb{H}^3$, and this line determines the matrix up to sign. According to [4, p. 63], there exists a natural one-to-one correspondence between line matrices and oriented lines in $\mathbb{H}^3$. Hereby, if a line matrix A determines an oriented line $\lambda_A = [e, e']$ with end points e and e', then the line matrix $-A$ determines the line $[e', e]$. Moreover, if a matrix $F \in SL(2, \mathbb{C})$ is considered as a motion of $\mathbb{H}^3$, then the matrix FAF^{-1} determines the line $[F(e), F(e')]$.

Definition (2.1.2). Let λ_A and λ_B be oriented lines determined by the line matrices A and B. A complex number μ is called a *complex distance* from λ_A to λ_B if its real part $\Re\,\mu$ is the distance from λ_A to λ_B, and its imaginary part $\Im\,\mu$ is the angle from λ_A to λ_B chosen in $[0, 2\pi)$.

We have [4, p. 68]

$$\cosh\mu = -\frac{1}{2}\,\mathrm{tr}\,(AB). \tag{2.1.3}$$

From now on, all lines in this paper will be assumed to be oriented.

Any isometry A of $\mathbb{H}^3$ which is neither parabolic nor the identity has two fixed points u and v in $\widehat{\mathbb{C}}$. It acts as a translation of distance r_A along the axis $\lambda_A = [u, v]$ and rotation of φ_A about λ_A.

Definition (2.1.4). We call *displacement* of A the complex number $\delta(A) = r_A + i\varphi_A$.

The isometry A, without an orientation of its axis, determines $\delta(A)$ up to sign. By [4, p. 46], for the isometry given by a matrix $A \in SL(2, \mathbb{C})$ we have

$$2\cosh\delta(A) = \mathrm{tr}\,(A^2) = \mathrm{tr}^{\,2}(A) - 2.$$

We remark that if $\delta(A) \neq 0$ then A has two different fixed points, so it admits an axis determined by these points. The line matrix $\widetilde{A}$ of this axis is defined by

$$\widetilde{A} = \frac{A - A^{-1}}{2i\sinh\frac{\delta(A)}{2}} \tag{2.1.5}$$

Since $\delta(A^{-1}) = -\delta(A)$, the matrices A and A^{-1} define the same line matrix $\widetilde{A} = \widetilde{A^{-1}}$ (see [4]).

Definition (2.1.6). The *complex length* γ_j of a singular component Σ^j of the cone-manifold C is the displacement $\delta(L_j)$ of the isometry L_j, where $L_j = h(l_j)$ is represented by the longitude l_j of Σ^j.

Immediately from the definition we get [4, p. 46]

$$2\cosh\gamma_j = \mathrm{tr}\,(L_j^2). \tag{2.1.7}$$

We note [2, p. 38] that the meridian-longitude pair $\{m_j,\, l_j\}$ of the oriented link is uniquely determined up to a common conjugating element of the group Φ. Hence, the complex length $\gamma_j = r_j + i\,\varphi_j$ is uniquely determined (mod $2\pi i$), up to a sign, by the above definition.

We need two conventions to correctly choose real and imaginary parts of γ_j. The first convention is the following. By the assumptions on the singular set we have $r_j \neq 0$. Hence, we can choose γ_j in such a way that $r_j > 0$. The second convention concerns the imaginary part φ_j. We want to choose φ_j so that the following identity holds

$$\cosh\frac{\gamma_j}{2} = -\frac{1}{2}\,\mathrm{tr}\,(L_j) \tag{2.1.8}$$

By virtue of the identity $\mathrm{tr}^{\,2}(L_j) - 2 = \mathrm{tr}\,(L_j^2)$, the equality (2.1.7) is a consequence of (2.1.8), but the converse, in general, is true only up to a sign. Under the second convention (2.1.7) and (2.1.8) are equivalent. The two above conventions lead to convenient analytic formulas in order to calculate γ_j and r_j. More precisely, there are simple relations between these numbers and the eigenvalues of the matrix L_j. Recall that $\det(L_j) = 1$. Since L_j is loxodromic, it has two eigenvalues f_j and $1/f_j$. We choose f_j so that $|f_j| > 1$. The case $|f_j| = 1$ is impossible because in this case the matrix L_j is elliptic and therefore $r_j = 0$. Hence

$$f_j = -e^{\frac{\gamma_j}{2}},\ \ |f_j| = e^{\frac{r_j}{2}}\,.$$

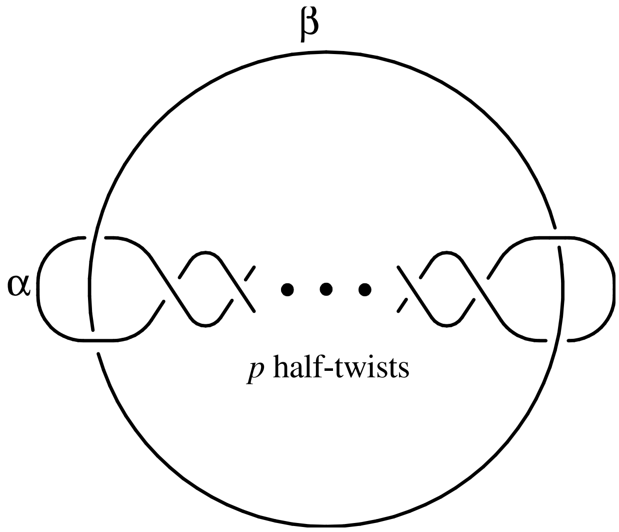

FIGURE 1. The cone-manifold $W_p(\alpha,\beta)$.

In this paper we consider a family of cone-manifolds whose singular sets are links which are generalizations of the Whitehead link. The link W_p, $p \geq 0$, is the two-component link depicted in Figure 1, where p is the number of half twists of one component. For this reason we will call them *twist links*. It is easy to see that W_0 is the torus link of type $(2,4)$ and W_1 is the Whitehead link. All twist links are two-bridge links, in particular W_p is the two-bridge link with slope $(4p+4)/(2p+1)$, for all $p \geq 0$. They are all hyperbolic, except for W_0.

Denote by $W_p(\alpha,\beta)$ the cone-manifold whose underlying space is the 3-sphere and whose singular set consists of the twist link W_p with cone angles $\alpha = 2\pi/m$ and $\beta = 2\pi/n$ (see Figure 1). It follows from Thurston's theorem that $W_p(\alpha,\beta)$, with $p \neq 0$, admits a hyperbolic structure for all sufficiently small α and β.

By Kojima's rigidity theorem [13] the hyperbolic structure is unique, up to isometry, if $0 \leq \alpha, \beta \leq \pi$.

In our paper we deal only with this range of angles.

Let us investigate the hyperbolic structure of the cone-manifold $W_p(\alpha,\beta)$. Its singular set $\Sigma = \Sigma^1 \cup \Sigma^2$ of consists of two components $\Sigma^1 = \Sigma^1(\alpha)$ and $\Sigma^2 = \Sigma^2(\beta)$ with cone-angles α and β respectively. $W_p(\alpha,\beta)$ defines a nonsingular but incomplete hyperbolic manifold $\mathcal{M} = W_p(\alpha,\beta) - \Sigma$. The fundamental group of the manifold $\mathcal{M}$ has the following presentation

$$\Phi_p = \langle s, t \mid \ sl_s = l_s s\rangle = \langle s, t \mid \ tl_t = l_t t\rangle\,,$$

where s and t (resp. l_s and l_t) are meridians (resp. longitudes) of the components Σ^1 and Σ^2 respectively.

We use the following expression of l_s in terms of s and t:

$$l_s = [s,t]^{\frac{p+1}{2}}[s,t^{-1}]^{\frac{p+1}{2}}, \quad \text{if } p \text{ is odd}, \tag{2.1.9}$$

$$l_s = s^{-1}[t,s]^{\frac{p}{2}}tst[s^{-1},t^{-1}]^{\frac{p}{2}}, \quad \text{if } p \text{ is even}\,, \tag{2.1.10}$$

where $[s,t] = sts^{-1}t^{-1}$.

The expressions for l_t can be easily obtained by exchanging s and t in the previous formulae.

Let

$$\hat{h} = \hat{h}_{\alpha,\beta} : \Phi_p \to PSL(2,\mathbb{C})$$

and

$$h = h_{\alpha,\beta} : \Phi_p \to SL(2,\mathbb{C})$$

be holonomy homomorphisms and $\Gamma_{\alpha,\beta} = h_{\alpha,\beta}(\Phi_p)$. The images $\hat{h}_{\alpha,\beta}(s)$ and $\hat{h}_{\alpha,\beta}(t)$ of s and t are rotations in $\mathbb{H}^3$ of angles α and β, respectively. The group $\Gamma_{\alpha,\beta}$ is generated by the two matrices $S = h_{\alpha,\beta}(s)$ and $T = h_{\alpha,\beta}(t)$ with the following properties:

$$\mathrm{tr}\,(S) = 2\cos\frac{\alpha}{2}, \quad \mathrm{tr}\,(T) = 2\cos\frac{\beta}{2}, \qquad SL_S = L_S S,$$

where $L_S = h_{\alpha,\beta}(l_s)$.

(2.2) Complex distance equation for two-bridge links. The fundamental group of (the exterior of) a link K is generated by two meridians if and only if K is a two-bridge link [1]. Moreover, a two-bridge link is hyperbolic if and only if its slope is different from $p/1$ and $p/(p-1)$ (see [21]).

PROPOSITION (2.2.1). *Let $\Phi = \langle s, t\rangle$ be the fundamental group of a hyperbolic two-bridge link K generated by the two meridians s and t. Let $\Gamma_{\alpha,\beta} = h_{\alpha,\beta}(\Phi)$ be the image of Φ under the holonomy homomorphism of the hyperbolic cone* *manifold $K(\alpha,\beta)$. Then, up to conjugation in $SL(2,\mathbb{C})$, the generators $S = h_{\alpha,\beta}(s)$ and $T = h_{\alpha,\beta}(t)$ of $\Gamma_{\alpha,\beta}$ can be chosen in such a way that*

$$(2.2.2)\quad S = \begin{pmatrix} \cos\frac{\alpha}{2} & i\,e^{\frac{\rho}{2}}\sin\frac{\alpha}{2} \\ i\,e^{-\frac{\rho}{2}}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}, \quad T = \begin{pmatrix} \cos\frac{\beta}{2} & i\,e^{-\frac{\rho}{2}}\sin\frac{\beta}{2} \\ i\,e^{\frac{\rho}{2}}\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix},$$

where ρ is the complex distance between the axis of S and T.

Proof. After a suitable conjugation in the group $SL(2,\mathbb{C})$, one can assume that the oriented axes of the elliptic elements S and T are $\lambda_S = [-e^{\frac{\rho}{2}}, e^{\frac{\rho}{2}}]$ and $\lambda_T = [-e^{-\frac{\rho}{2}}, e^{-\frac{\rho}{2}}]$. Since $\mathrm{tr}\,(S) = 2\cos\frac{\alpha}{2}$ and $\mathrm{tr}\,(T) = 2\cos\frac{\beta}{2}$, the matrices S and T are given by (2.2.2). Check that ρ coincides with the complex distance $\rho(S,T)$ between λ_S and λ_T. The line matrices $\widetilde{S}$ and $\widetilde{T}$, corresponding to these axes, can be obtained by (2.1.5). Since $\delta(S) = i\,\alpha$ and $\delta(T) = i\,\beta$, we have $\widetilde{S} = \begin{pmatrix} 0 & -ie^{\frac{\rho}{2}} \\ -ie^{-\frac{\rho}{2}} & 0 \end{pmatrix}$ and $\widetilde{T} = \begin{pmatrix} 0 & -ie^{-\frac{\rho}{2}} \\ -ie^{\frac{\rho}{2}} & 0 \end{pmatrix}$ respectively. By [4, p. 68] we get $\cosh\rho(S,T) = -\frac{1}{2}\,\mathrm{tr}\,(\widetilde{S}\widetilde{T}) = \cosh\rho$. $\square$

The following two propositions can be obtained by direct calculation from the above statement.

Proposition (2.2.3). *Let*

$$\Phi_2 = \langle s,t : sl = ls,\ l = s^{-1}tst^{-1}s^{-1}tsts^{-1}t^{-1}st\rangle$$

be the fundamental group of the two-bridge link W_2 with slope $12/5$ and $\Gamma_{\alpha,\beta} = h_{\alpha,\beta}(\Phi_2) = \langle S,T\rangle$ be the image of Φ_2 under the holonomy homomorphism of the hyperbolic cone manifold $W_2(\alpha,\beta)$. Denote by $\rho = \rho(S,T)$ the complex distance between the axes of $S = h_{\alpha,\beta}(s)$ and $T = h_{\alpha,\beta}(t)$. Then $u = \cosh\rho$ is a non-real root of the complex distance equation

$$(2.2.4)\quad 4z^3 - 4abz^2 + (3a^2b^2 + 3a^2 + 3b^2 - 1)z - ab(a^2b^2 + a^2 + b^2 - 3) = 0,$$

where $a = \cot\frac{\alpha}{2}$ and $b = \cot\frac{\beta}{2}$.

Proof. Denote by $L = S^{-1}TST^{-1}S^{-1}TSTS^{-1}T^{-1}ST$ the image of the longitude l under the holonomy homomorphism $h = h_{\alpha,\beta} : \Phi_2 \to SL(2,\mathbb{C})$. Then we have $SL = LS$.

Let N be a line matrix corresponding to the common normal to the axes of S and T. If S and T are represented in the form (2.2.2) then one can take $N = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. It is not difficult to verify that $NSN^{-1} = S^{-1}$ and $NTN^{-1} = T^{-1}$.

To complete the proof, we need the following lemma, which gives simple criteria for matrices S and L to be permutable.

Lemma (2.2.5). *The following conditions are equivalent: (i) $SL = LS$; (ii) $NLN^{-1} = L^{-1}$; (iii) $\mathrm{tr}\,(NL) = 0$.*

Proof. First we show that (i) and (ii) are equivalent. Indeed, since $L = S^{-1}TST^{-1}S^{-1}TSTS^{-1}T^{-1}ST$ we have

$$NLN^{-1} = ST^{-1}S^{-1}TST^{-1}S^{-1}T^{-1}STS^{-1}T^{-1} = SL^{-1}S^{-1}\,.$$

Hence (ii) holds if and only if S and L^{-1} commute. The last property is equivalent to (i). Because of $N^2 = -I$ the condition (ii) can be rewritten in the form $NLNL = -I$; this is equivalent to (iii). $\square$

By this lemma and direct calculation we have

$$\operatorname{tr}(NL) = \frac{-4i\sinh\rho}{(1+a^2)^3(1+b^2)^3}\cdot(4u^2+a^2b^2+a^2+b^2-3)\cdot$$
$$\cdot(4u^3-4abu^2+(3a^2b^2+3a^2+3b^2-1)u-ab(a^2b^2+a^2+b^2-3))=0,$$

where $u=\cosh\rho$.

Now we have to show that u is a non-real root of (2.2.4). Since $\Gamma_{\alpha,\beta}$ is the holonomy group of a hyperbolic cone-manifold, it is non-elementary[1] and is not conjugate to a subgroup of $SL(2,\mathbb{R})$ [8].

If $\sinh\rho = 0$ then the axes S and T coincide, and the group $\Gamma_{\alpha,\beta}$ is elementary.

If u is a root of equation

$$4u^2+a^2b^2+a^2+b^2-3=0$$

then by the equality

$$\operatorname{tr}L-2=-\frac{4(a^2+u^2)(4u^2+a^2b^2+a^2+b^2-3)^2}{(a^2+1)^3(b^2+1)^3}$$

we have $\operatorname{tr}L=2$. From (2.1.8) we obtain

$$\cosh\frac{\gamma_S}{2}=-\frac{1}{2}\operatorname{tr}(L)=-1.$$

Hence $\gamma_S=r_S+i\varphi_S=2\pi i$ and the real length r_S of the link component Σ_1 is equal to zero, which is a contradiction.

Suppose that $u=\cosh\rho$ is a real root. Let

$$R(z_1,z_2,z_3,z_4)=\frac{(z_3-z_1)(z_4-z_2)}{(z_3-z_2)(z_4-z_1)}$$

be the cross ratio of the four points $z_1,z_2,z_3,z_4\in\widehat{\mathbb{C}}$. Then

$$R(-e^{\frac{\rho}{2}},e^{\frac{\rho}{2}},-e^{-\frac{\rho}{2}},e^{-\frac{\rho}{2}})=(\cosh\rho-1)/(\cosh\rho+1)\in\mathbb{R}\cup\{\infty\}.$$

We have that the axes $[-e^{\frac{\rho}{2}},e^{\frac{\rho}{2}}]$ and $[-e^{-\frac{\rho}{2}},e^{-\frac{\rho}{2}}]$ of S and T lie in a common plane. If the axes intersect then the group $\Gamma_{\alpha,\beta}=\langle S,T\rangle$ has a fixed point and is elementary. If they do not intersect, $\Gamma_{\alpha,\beta}$ is conjugate to a subgroup of $SL(2,\mathbb{R})$.

Therefore, we have shown that u is a non-real root of (2.2.4) and the proof of Proposition (2.2.3) is complete. $\square$

The next proposition can be proved by similar arguments.

Proposition (2.2.6). *Let*

$$\Phi_3=\langle s,t:\ sl\ =\ ls,\ l\ =\ sts^{-1}t^{-1}sts^{-1}t^{-1}st^{-1}s^{-1}tst^{-1}s^{-1}t\rangle$$

be the fundamental group of the two-bridge link W_3 *with the slope* $16/7$ *and* $\Gamma_{\alpha,\beta}=h_{\alpha,\beta}(\Phi_3)=\langle S,T\rangle$ *the image of* Φ_3 *under the holonomy homomorphism of a hyperbolic cone manifold* $W_3(\alpha,\beta)$ *generated by* $S=h_{\alpha,\beta}(s)$ *and* $T=h_{\alpha,\beta}(t)$.

[1] A subgroup G of $SL(2,\mathbb{C})$ is called *elementary* if it has a finite orbit in $\mathbb{H}^3\cup\widehat{\mathbb{C}}$.

Denote by $\rho = \rho(S,T)$ the complex distance between the axes of S and T. Then $u = \cosh\rho$ is a non-real root of the complex distance equation

$$0 = 8u^5 + 8abu^4 + 8(a^2b^2 + a^2 + b^2 - 1)u^3 + 4ab(a^2b^2 + a^2 + b^2 - 3)u^2 +$$

$$(a^4b^4 + 2a^4b^2 + 2a^2b^4 - 4a^2b^2 + a^4 + b^4 - 6a^2 - 6b^2 + 1)u - 4ab(a^2b^2 + a^2 + b^2 - 1),$$

where $a = \cot\frac{\alpha}{2}$ and $b = \cot\frac{\beta}{2}$.

(2.3) Tangent, Sine and Cosine rules. If we set $z = \mathrm{tr}\,(S^{-1}T)$ then, from the presentation in Proposition (2.2.1), we have

$$z = 2(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + u\sin\frac{\alpha}{2}\sin\frac{\beta}{2}),$$

where $u = \cosh\rho$.

The algebraic equation for z and its behaviour was considered in a number of papers (see [3], [5], [8] and others) devoted to $PSL(2,\mathbb{C})$ representations of two-generator groups.

In general, the equation for u (as well as for z) is very complicated, even for twist links. In spite of this, since $u = \cosh\rho$ has a very clear geometric sense, we are able to produce some general results for twist links without calculating u.

Proposition (2.3.1). *Let $W_p(\alpha,\beta)$ be a hyperbolic twist link cone-manifold. Denote by $S = h_{\alpha,\beta}(s)$ and $T = h_{\alpha,\beta}(t)$ the images of the generators of the group $\Phi_p = \langle s,t \mid sl_s = l_s s\rangle$ under the holonomy homomorphism $h_{\alpha,\beta} : \Phi_p \to SL(2,\mathbb{C})$. Set $u = \cosh\rho$, where ρ is the complex distance between the axes of S and T, such that $\Im u > 0$. Moreover, denote by γ_α and γ_β the complex lengths of the singular components of $W_p(\alpha,\beta)$ with cone-angles α and β respectively. Then*

$$u = i\,\cot\frac{\alpha}{2}\coth\frac{\gamma_\beta}{4} = i\,\cot\frac{\beta}{2}\coth\frac{\gamma_\alpha}{4}.$$

Proof. To prove the statement we need to calculate the complex distance between axes of elliptic elements S and T in two ways. By definition, $L_S = h_{\alpha,\beta}(l_s)$ and $L_T = h_{\alpha,\beta}(l_t)$, where l_s and l_t are the longitudes of the singular components of $W_p(\alpha,\beta)$ with cone angles α and β, respectively.

First of all, we fix an orientation on the axes of S and T by the following line matrices

$$\widetilde{S} = \frac{S - S^{-1}}{2\,i\,\sinh\frac{i\alpha}{2}}, \qquad \widetilde{T} = \frac{T - T^{-1}}{2\,i\,\sinh\frac{i\beta}{2}}.$$

Then the complex distance $\rho(S,T)$ between the oriented axes of S and T is defined by (2.1.3):

$$\cosh\rho(S,T) = -\frac{1}{2}\,\mathrm{tr}\,(\widetilde{S}\widetilde{T}).$$

Using (2.1.5) we define the line matrices for L_S and L_T as

$$\widetilde{L_S} = \frac{L_S - L_S^{-1}}{2i\sinh\frac{\gamma_\alpha}{2}}, \qquad \widetilde{L_T} = \frac{L_T - L_T^{-1}}{2i\sinh\frac{\gamma_\beta}{2}}.$$

To continue the proof, we need two lemmas:

Lemma (2.3.2). *For every S,T we have $\widetilde{S} = -\widetilde{L_S}$ and $\widetilde{T} = -\widetilde{L_T}$.*

Proof. Up to conjugation in $SL(2,\mathbb{C})$, we can assume that S is given by

$$S = \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{pmatrix}.$$

Note that L_S is a loxodromic element, with displacement γ_α, and commutes with S. Since $\widetilde{L_S^{-1}} = \widetilde{L_S}$, we can assume that

$$L_S = \begin{pmatrix} \pm e^{\frac{\gamma_\alpha}{2}} & 0 \\ 0 & \pm e^{-\frac{\gamma_\alpha}{2}} \end{pmatrix}$$

By convention (see formula (2.1.8)) we have

$$\mathrm{tr}\,(L_S) = -2\cosh\frac{\gamma_\alpha}{2}.$$

Hence

$$L_S = \begin{pmatrix} -e^{\frac{\gamma_\alpha}{2}} & 0 \\ 0 & -e^{-\frac{\gamma_\alpha}{2}} \end{pmatrix}$$

and we obtain

$$\widetilde{L}_S = \frac{L_S - L_S^{-1}}{2i\sinh\frac{\gamma_\alpha}{2}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and

$$\widetilde{S} = \frac{S - S^{-1}}{2\,i\sinh\frac{i\,\alpha}{2}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

$\square$

Lemma (2.3.3). *For every S,T we have $\mathrm{tr}\,(S) = \mathrm{tr}\,(S^{-1}L_T)$ and $\mathrm{tr}\,(T) = \mathrm{tr}\,(T^{-1}L_S)$.*

Proof. To prove $\mathrm{tr}\,(T) = \mathrm{tr}\,(T^{-1}L_S)$ it is enough to show that $T^{-1}L_S$ is conjugate to either T or T^{-1} in the group $\Gamma_{\alpha,\beta}$. If p is odd, we have from (2.1.9):

$$T^{-1}L_S = T^{-1}[S,T]^{\frac{p+1}{2}}\,[S,T^{-1}]^{\frac{p+1}{2}} = [T^{-1},S]^{\frac{p+1}{2}}\,T^{-1}[T^{-1},S]^{-\frac{p+1}{2}}.$$

If p is even, we have from (2.1.10):

$$T^{-1}L_S = T^{-1}S^{-1}[T,S]^{\frac{p}{2}}\,TST\,[S^{-1},T^{-1}]^{\frac{p}{2}} = T^{-1}S^{-1}[T,S]^{\frac{p}{2}}\,T\,[T,S]^{-\frac{p}{2}}\,ST.$$

The equality $\mathrm{tr}\,(S) = \mathrm{tr}\,(S^{-1}L_T)$ can be obtained in a similar way. $\square$

To complete the proof of Proposition (2.3.1), we note that $\mathrm{tr}\,(XY) = \mathrm{tr}\,(X)\,\mathrm{tr}\,(Y) - \mathrm{tr}\,(X^{-1}Y)$, $\mathrm{tr}\,(X^{-1}) = \mathrm{tr}\,(X)$ and $\mathrm{tr}\,(XY) = \mathrm{tr}\,(X^{-1}Y^{-1})$ holds for all $X,Y \in SL(2,\mathbb{C})$. By Lemma (2.3.2), Lemma (2.3.3) and formulae $\mathrm{tr}\,(S) = 2\cos\frac{\alpha}{2}$, $\mathrm{tr}\,(L_S) = -2\cosh\frac{\gamma_\alpha}{2}$, we have

$$\cosh\rho(S,T) = -\frac{1}{2}\,\mathrm{tr}\,(\widetilde{S}\widetilde{T}) = \frac{1}{2}\,\mathrm{tr}\,(\widetilde{S}\widetilde{L_T}) =$$

$$= \frac{1}{2}\,\mathrm{tr}\left(\frac{S-S^{-1}}{2\sin\frac{\alpha}{2}}\,\frac{L_T - L_T^{-1}}{2i\sinh\frac{\gamma_\beta}{2}}\right) = \frac{\mathrm{tr}\,(SL_T - S^{-1}L_T - SL_T^{-1} + S^{-1}L_T^{-1})}{8i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} =$$

$$= \frac{2(\mathrm{tr}\,(SL_T) - \mathrm{tr}\,(S^{-1}L_T))}{8i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} = \frac{\mathrm{tr}\,(S)\mathrm{tr}\,(L_T) - 2\mathrm{tr}\,(S^{-1}L_T)}{4i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} =$$

$$= \frac{\mathrm{tr}\,(S)\mathrm{tr}\,(L_T) - 2\mathrm{tr}\,(S)}{4i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} = \frac{\mathrm{tr}\,(S)(2-\mathrm{tr}\,(L_T))}{-4i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}}$$

$$= \frac{2\cos\frac{\alpha}{2}(2+2\cosh\frac{\gamma_\beta}{2})}{-4i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\beta}{2}} = i\cot\frac{\alpha}{2}\coth\frac{\gamma_\beta}{4}.$$

Since $\cosh\rho(S,T) = \cosh\rho(T,S) = u$ the statement follows. □

As an immediate consequence of the previous proposition, we have the following result.

THEOREM (2.3.4). (The Tangent Rule). *Suppose that $W_p(\alpha,\beta)$ is a hyperbolic cone-manifold. Denote by γ_α and γ_β complex lengths of the singular geodesics of $W_p(\alpha,\beta)$ with cone angles α and β respectively. Then*

$$\frac{\tanh\frac{\gamma_\alpha}{4}}{\tanh\frac{\gamma_\beta}{4}} = \frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}}.$$

The following two theorems are consequences of the Tangent Rule.

THEOREM (2.3.5). *(The Sine Rule) Let $\gamma_\alpha = r_\alpha + i\,\varphi_\alpha$ and $\gamma_\beta = r_\beta + i\,\varphi_\beta$ be the complex lengths of the singular geodesics of a hyperbolic cone-manifold $W_p(\alpha,\beta)$ with cone angle α and β respectively. Then*

$$\frac{\sin\frac{\varphi_\alpha}{2}}{\sinh\frac{r_\alpha}{2}} = \frac{\sin\frac{\varphi_\beta}{2}}{\sinh\frac{r_\beta}{2}}.$$

Proof. By the Tangent Rule we have

$$\frac{\tanh\frac{\gamma_\alpha}{4}}{a} = \frac{\tanh\frac{\gamma_\beta}{4}}{b},$$

where $a = \tan\frac{\alpha}{2}$ and $B = \tan\frac{\beta}{2}$ are real numbers. Hence

$$\frac{\Re(\tanh\frac{\gamma_\alpha}{4})}{a} = \frac{\Re(\tanh\frac{\gamma_\beta}{4})}{b},$$

and

$$\frac{\Im(\tanh\frac{\gamma_\alpha}{4})}{a} = \frac{\Im(\tanh\frac{\gamma_\beta}{4})}{b}.$$

Dividing one equation by the other we obtain

$$\frac{\Re(\tanh\frac{\gamma_\alpha}{4})}{\Im(\tanh\frac{\gamma_\alpha}{4})} = \frac{\Re(\tanh\frac{\gamma_\beta}{4})}{\Im(\tanh\frac{\gamma_\beta}{4})}.$$

By direct calculations we have

$$\Re(\tanh\frac{\gamma_\alpha}{4}) = \frac{1}{2}(\tanh\frac{\gamma_\alpha}{4} + \tanh\frac{\bar\gamma_\alpha}{4}) = \frac{\sinh\frac{r_\alpha}{2}}{\cosh\frac{r_\alpha}{2} + \cos\frac{\varphi_\alpha}{2}}$$

and

$$\Im(\tanh\frac{\gamma_\alpha}{4}) = \frac{1}{2i}(\tanh\frac{\gamma_\alpha}{4} - \tanh\frac{\bar\gamma_\alpha}{4}) = \frac{\sin\frac{\varphi_\alpha}{2}}{\cosh\frac{r_\alpha}{2} + \cos\frac{\varphi_\alpha}{2}}.$$

Since $r_\alpha > 0$, we have $\cosh\dfrac{r_\alpha}{2} > 1$. Therefore $\cosh\dfrac{r_\alpha}{2} + \cos\dfrac{\varphi_\alpha}{2} > 0$ and the result follows. □

THEOREM (2.3.6). *(The Cosine Rule) Let $\gamma_\alpha = r_\alpha + i\,\varphi_\alpha$ and $\gamma_\beta = r_\beta + i\,\varphi_\beta$ be the complex lengths of the singular geodesics of a hyperbolic cone-manifold $W_p(\alpha,\beta)$ with cone angle α and β respectively. Then*

$$\frac{\cos\frac{\varphi_\alpha}{2}\cosh\frac{r_\beta}{2}-\cos\frac{\varphi_\beta}{2}\cosh\frac{r_\alpha}{2}}{\cosh\frac{r_\alpha}{2}\cosh\frac{r_\beta}{2}-\cos\frac{\varphi_\alpha}{2}\cos\frac{\varphi_\beta}{2}}=\frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}.$$

Proof. By the Tangent Rule

$$\frac{\tanh\frac{\gamma_\alpha}{4}\tanh\frac{\bar\gamma_\alpha}{4}}{a^2}=\frac{\tanh\frac{\gamma_\beta}{4}\tanh\frac{\bar\gamma_\beta}{4}}{b^2},$$

where $a=\tan\frac{\alpha}{2}$ and $b=\tan\frac{\beta}{2}$. Hence

$$\frac{1+\cos\alpha}{1-\cos\alpha}\,\frac{\cosh\frac{r_\alpha}{2}-\cos\frac{\varphi_\alpha}{2}}{\cosh\frac{r_\alpha}{2}+\cos\frac{\varphi_\alpha}{2}}=\frac{1+\cos\beta}{1-\cos\beta}\,\frac{\cosh\frac{r_\beta}{2}-\cos\frac{\varphi_\beta}{2}}{\cosh\frac{r_\beta}{2}+\cos\frac{\varphi_\beta}{2}}.$$

Set

$$p=\cos\alpha,\quad q=\cos\beta,\quad p'=\frac{\cos\frac{\varphi_\alpha}{2}}{\cosh\frac{r_\alpha}{2}},\quad q'=\frac{\cos\frac{\varphi_\beta}{2}}{\cosh\frac{r_\beta}{2}}$$

and rewrite the above equation in the form

$$\frac{1+p}{1-p}\,\frac{1-p'}{1+p'}=\frac{1+q}{1-q}\,\frac{1-q'}{1+q'},$$

or, equivalently, as

$$\log\frac{1+p}{1-p}+\log\frac{1-p'}{1+p'}=\log\frac{1+q}{1-q}+\log\frac{1-q'}{1+q'}.$$

Since $\operatorname{arctanh} p=\dfrac{1}{2}\log\dfrac{1+p}{1-p}$ we have

$$\operatorname{arctanh} p-\operatorname{arctanh} p'=\operatorname{arctanh} q-\operatorname{arctanh} q'.$$

and

$$\operatorname{arctanh} p-\operatorname{arctanh} q=\operatorname{arctanh} p'-\operatorname{arctanh} q'.$$

Hence

$$\frac{p-q}{1-pq}=\frac{p'-q'}{1-p'q'}$$

and, after substituting the expressions for p,q,p',q' in the last formula, we obtain the desired identity. □

We remark that, in the case of Whitehead link cone-manifolds, Tangent and Sine rules were obtained in [14].

3. Explicit volume calculation for twist link cone-manifolds

(3.1) The Schläfli formula. In this section we will obtain explicit formulae for the volume of some special cone-manifolds in the hyperbolic and spherical geometries. In the case of complete hyperbolic structure on the simplest knot and link complements such formulas, in terms of the Lobachevsky function, are well-known and widely represented in [21]. In general, a hyperbolic cone-manifold can be obtained by completion of a non-complete hyperbolic structure on a suitable knot or link complement. If the cone-manifold is compact, explicit formulas are only known in a few cases [9], [10], [11], [15], [16], [17], [18], [19]. In all these cases the starting point for the volume calculation is the Schläfli formula (see, for example [11]).

THEOREM (3.1.1). *(The Schläfli volume formula) Suppose that C_t is a smooth 1-parameter family of (curvature K) cone-manifold structures on an n-manifold, with singular locus Σ of a fixed topological type. Then the derivative of volume of C_t satisfies*

$$(n-1)K dV(C_t) = \sum_{\sigma} V_{n-2}(\sigma)\, d\theta(\sigma)$$

where the sum is over all the components σ of the singular locus Σ, and $\theta(\sigma)$ is the cone angle along σ.

In the present paper we will deal mostly with three-dimensional cone-manifold structures of negative constant curvature $K = -1$. The Schläfli formula in this case reduces to

$$dV = -\frac{1}{2}\sum_{i} r_i d\theta_i,$$

where the sum is taken over all the components of the singular set Σ with lengths r_i and cone angles θ_i.

Our aim is to obtain the volume formulas for twist link hyperbolic cone-manifolds $W_2(\alpha,\beta)$. We note that the volume formula for $W_1(\alpha,\beta)$ were obtained earlier in [16] and [19].

PROPOSITION (3.1.2). *Let $W_2(\alpha,\beta)$ be a hyperbolic cone-manifold and r_α, r_β the lengths of its singular components, with cone angles α and β respectively. If $a = \cot\frac{\alpha}{2}$ and $b = \cot\frac{\beta}{2}$, then*

$$r_\alpha = 2i\ \arctan\frac{a}{\zeta} - 2i\ \arctan\frac{a}{\overline{\zeta}}, \tag{3.1.3}$$

$$r_\beta = 2i\ \arctan\frac{b}{\zeta} - 2i\ \arctan\frac{b}{\overline{\zeta}}, \tag{3.1.4}$$

where ζ is a root of the equation

$$4(z^2+a^2)(z^2+b^2) - (1+a^2)(1+b^2)(z-z^2)^2 = 0, \tag{3.1.5}$$

with $\Im(\zeta) > 0$.

Proof. By Proposition (2.3.1) we have

$$i\, b \coth\frac{\gamma_\alpha}{4} = i\, a \coth\frac{\gamma_\beta}{4} = u, \tag{3.1.6}$$

where $u = \cosh\rho$ and ρ is a complex distance between the axes of S and T, chosen so that $\Im u > 0$. By Proposition (2.2.3), u is a root of the cubic equation

$$4z^3 - 4abz^2 + (3a^2b^2 + 3a^2 + 3b^2 - 1)z - ab(a^2b^2 + a^2 + b^2 - 3) = 0\,.$$

From (3.1.6), for a suitable choice of analytical branches,

$$r_\alpha = \frac{\gamma_\alpha}{2} + \frac{\overline{\gamma}_\alpha}{2} = 2i\ \arctan\frac{\overline{u}}{b} - 2i\ \arctan\frac{u}{b} = 2i\ \arctan\frac{a}{\zeta} - 2i\ \arctan\frac{a}{\overline{\zeta}}\,,$$

where $\zeta = ab/\overline{u}$, $\Im(\zeta) > 0$, satisfies the equation

$$Q(z) = (a^2b^2 + a^2 + b^2 - 3)z^3 - (3a^2b^2 + 3a^2 + 3b^2 - 1)z^2 + 4a^2b^2z - 4a^2b^2 = 0\,.$$

To finish the proof we note that

$$(z+1)Q(z) = -4(z^2+a^2)(z^2+b^2) + (1+a^2)(1+b^2)(z-z^2)^2.$$

□

In the next section we will apply this result to calculate the volume of $W_2(\alpha,\beta)$ via the Schläfli formula.

We remark that formulae (3.1.3) and (3.1.4), as a consequence of the Tangent Rule, also hold for all twist links W_p, with $\zeta = ab/\bar{u}$, where $u = \cosh\rho$.

For example, an analog for the algebraic equation (3.1.5), in the case of twist link W_3, can easily be obtained from Proposition (2.2.6). But in this case the equation become too complicated, and we are not able to explicitly find the integrand in the Schläfli formula.

(3.2) Volume of twist link cone-manifolds. The case of the Whitehead link cone manifolds $W_1(\alpha,\beta)$ has already been solved (see [16] and [19]).

Theorem (3.2.1). [16, 19] *Let $W_1(\alpha,\beta)$ be a hyperbolic Whitehead link cone-manifold with cone angles α and β. Then the volume of $W_1(\alpha,\beta)$ is given by the formula*

$$\operatorname{Vol} W_1(\alpha,\beta) = i\int_{\bar{\zeta}}^{\zeta} \log\left[\frac{2(z^2+a^2)(z^2+b^2)}{(1+a^2)(1+b^2)(z^2-z^3)}\right]\frac{dz}{z^2-1}.$$

where $a = \cot\frac{\alpha}{2}$, $b = \cot\frac{\beta}{2}$ and ζ is a non-real root, with $\Im(\zeta) > 0$, of the equation

$$2(z^2+a^2)(z^2+b^2) - (1+a^2)(1+b^2)(z^2-z^3) = 0.$$

The main result of this section is the following.

Theorem (3.2.2). *Let $W_2(\alpha,\beta)$ be a hyperbolic twist link cone-manifold with cone angles α and β. Then the volume of $W_2(\alpha,\beta)$ is given by the formula*

$$\operatorname{Vol} W_2(\alpha,\beta) = i\int_{\bar{\zeta}}^{\zeta} \log\left[\frac{4(z^2+a^2)(z^2+b^2)}{(1+a^2)(1+b^2)(z-z^2)^2}\right]\frac{dz}{z^2-1}. \tag{3.2.3}$$

where $a = \cot\frac{\alpha}{2}$, $b = \cot\frac{\beta}{2}$ and ζ is a non-real root, with $\Im(\zeta) > 0$, of the equation

$$4(z^2+a^2)(z^2+b^2) - (1+a^2)(1+b^2)(z-z^2)^2 = 0. \tag{3.2.4}$$

Proof. Denote by $V = \operatorname{Vol} W_2(\alpha,\beta)$ the hyperbolic volume of $W_2(\alpha,\beta)$. Then by virtue of the Schläfli formula we have

$$\frac{\partial V}{\partial \alpha} = -\frac{r_\alpha}{2},\quad \frac{\partial V}{\partial \beta} = -\frac{r_\beta}{2}, \tag{3.2.5}$$

where r_α and r_α are the lengths of the singular geodesics having cone angles α and β respectively.

We note that for $\alpha = \beta$ and $\Im(\zeta) \to 0$ the geometrical limit of the cone-manifold $W_2(\alpha,\,\alpha)$ is a Euclidean cone manifold $W_2(\alpha_0,\,\alpha_0)$, where $\alpha_0 = 2.7243... < \pi$. (See Example 1 in Section 3.3 below). Hence, by Theorem 7.1.2 of [13], we have

$$V \to 0 \text{ as } \alpha = \beta \text{ and } \Im(\zeta) \to 0. \tag{3.2.6}$$

We set
$$W = \int_{\overline{\zeta}}^{\zeta} F(z,a,b)\,dz,$$
where
$$F(z,a,b) = \frac{i}{z^2-1}\,\log\frac{4(z^2+a^2)(z^2+b^2)}{(1+a^2)(1+b^2)(z-z^2)^2}.$$
Now we show that W satisfies conditions (3.2.5) and (3.2.6). So $W = V$ and the theorem follows.

By the Leibniz formula we have

$$\frac{\partial W}{\partial\alpha} = F(\zeta,a,b)\frac{\partial\zeta}{\partial\alpha} - F(\overline{\zeta},a,b)\frac{\partial\overline{\zeta}}{\partial\alpha} + \int_{\overline{\zeta}}^{\zeta}\frac{\partial F(z,a,b)}{\partial a}\,\frac{\partial a}{\partial\alpha}\,dz \tag{3.2.7}$$

We note that $F(\overline{\zeta},a,b) = F(\zeta,a,b) = 0$ if $\overline{\zeta},\zeta,a$ and b are the same as in the statement of the theorem. Moreover, since $\alpha = 2\operatorname{arccot} a$ we have $\dfrac{\partial a}{\partial\alpha} = -\dfrac{1+a^2}{2}$ and

$$\frac{\partial F(z,a,b)}{\partial a}\,\frac{\partial a}{\partial\alpha} = -\frac{ia}{z^2+a^2}.$$

Hence, by Proposition (3.1.2), we obtain from (3.2.7)
$$\frac{\partial W}{\partial\alpha} = -ia\int_{\overline{\zeta}}^{\zeta}\frac{dz}{z^2+a^2} = -i\,\arctan\frac{a}{\zeta} + i\,\arctan\frac{a}{\overline{\zeta}} = -\frac{r_\alpha}{2}.$$

The equation $\dfrac{\partial W}{\partial\beta} = -\dfrac{r_\beta}{2}$ can be obtained in the same way. The boundary condition (3.2.6) for the function W follows from the integral formula. □

(3.3) Particular cases and examples.

1. Case $\alpha = \beta$. In this case, Equation (3.2.4) splits into two quadratic equations:
$$(1+a^2)(z-z^2) + 2(z^2+a^2) = 0$$
and
$$(1+a^2)(z-z^2) - 2(z^2+a^2) = 0.$$
The first has two real roots $z = -1$ and $z = 2a^2/(a^2-1)$. The second has two non-real roots
$$z_{1,2} = \frac{1+a^2 \pm \sqrt{1-22a^2-7a^4}}{2(3+a^2)}.$$
By [10], $\Delta = 1-22a^2-7a^4$ is < 0 in the hyperbolic case, $= 0$ in the Euclidean case and > 0 in the spherical case. In the Euclidean case we obtain $a^2 = \cot^2(\alpha_0/2) = (\sqrt{128}-11)/7 = 0.0448...$ and $a = a_0 = \cot(\alpha_0/2) = 0.2116...$. So the cone-manifold is hyperbolic for $0 \le \alpha < \alpha_0 = 2.7243...$ and is Euclidean for $\alpha = \alpha_0$.

From (3.2.3) we have
$$\operatorname{Vol} W_2(\alpha,\alpha) = i\int_{z_1}^{z_2}\log\left[\frac{2(z^2+a^2)}{(z-z^2)(1+a^2)}\right]^2\frac{dz}{z^2-1}.$$

By differentiation with respect to a and then by integration with respect to z we obtain

$$\operatorname{Vol} W_2(\alpha,\alpha) = 4\int_{a_0}^{a} \operatorname{arctanh} \frac{\sqrt{7t^4+22t^2-1}}{t(5+t^2)} \frac{dt}{t^2+1}.$$

Since the integrand is purely imaginary for $0 \le t < a_0$ we are able to compute the volume in the more convenient way

$$\operatorname{Vol} W_2(\alpha,\alpha) = 4\Re\int_{0}^{a} \operatorname{arctanh} \frac{\sqrt{7t^4+22t^2-1}}{t(5+t^2)} \frac{dt}{t^2+1},$$

where $a = \cot\frac{\alpha}{2}$.

2. Case $\alpha=\beta=\pi/2$. In this case equation (3.2.4) becomes

$$(z+1)(z^2-z+2)=0.$$

Hence, the non-real roots are

$$z_{1,2} = \frac{1 \pm i\sqrt{7}}{2}$$

and

$$\operatorname{Vol} W_2(\pi/2,\pi/2) = 2i\int_{\frac{1-i\sqrt{7}}{4}}^{\frac{1+i\sqrt{7}}{4}} \log\frac{z^2+1}{z-z^2}\frac{dz}{z^2-1} = 2.6667...$$

3. Case $\alpha=\beta=0$. Recall that $W_2(0,0)$ is the complete hyperbolic manifold $S^3 \setminus W_2$. By arguments similar to those of the previous case, we obtain

$$\operatorname{Vol} W_2(0,0) = 2i\int_{\frac{1-i\sqrt{7}}{2}}^{\frac{1+i\sqrt{7}}{2}} \log\frac{2}{z-z^2}\frac{dz}{z^2-1} = 5.3334...$$

Note that $\operatorname{Vol} W_2(0,0) = 2\operatorname{Vol} W_2(\pi/2,\pi/2)$.

4. Case $\alpha=0$, $\beta=\pi/3$. In this case equation (3.2.4) reduces to

$$(1+z)(3-3z+3z^2-z^3)=0.$$

Hence, the non-real roots are

$$z_{1,2} = 1 - \frac{1\pm i\sqrt{3}}{\sqrt[3]{4}}$$

and

$$\operatorname{Vol} W_2(0,\pi/3) = i\int_{1-\frac{1+i\sqrt{3}}{\sqrt[3]{4}}}^{1-\frac{1-i\sqrt{3}}{\sqrt[3]{4}}} \log\frac{z^2+3}{(z-z^2)^2}\frac{dz}{z^2-1} = 4.6165...$$

The results of the above numerical calculation coincide with the corresponding results obtained by Weeks's SnapPea program [22].

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